Large and moderate deviation principles for McKean-Vlasov SDEs with jumps

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- Wei Liu, Yulin Song, Jianliang Zhai, Tusheng Zhang, *Large and moderate deviation principles for McKean-Vlasov SDEs with jumps,* arXiv:2011.08403; Potential Analysis, 2022, published online, 1-50.

Mean-field interacting particle system

Consider the following interacting particle system:

$$dX_t^{i,N} = b_t(X_t^{i,N}, L_t^N)dt + \sigma_t(X_t^{i,N}, L_t^N)dB_t^i, 1 \leq i \leq N$$

- $L_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ empirical measure
- coefficients

$$b : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$$
$$\sigma : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times m},$$

  where $\mathcal{P}(\mathbb{R}^d)$ is the space of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, endowed with the weak convergence topology.
- $B^1, \cdots, B^N$ independent Brownian motions taking values in $\mathbb{R}^d$
- $X_0^{1,N}, \cdots, X_0^{N,N}$ i.i.d., and independent of the Brownian motions $B^1, \cdots, B^N$

Applications

Mean-field interacting particle systems have been extensively studied in recent 40 years due to their wide range of applications in several fields including *physics, chemistry, biology, economics, mean-field games, financial mathematics, social science, machine learning* and so on.

- Physics, Chemistry: ions and electrons in plasmas, molecules in a fluid, galaxies in large scale cosmological models
- Biology: collective behaviors, neuronal network
- Economics, finances and Social Science: opinion dynamics, consensus model, mean field games
- Machine learning: deep learning, artificial neural network, distribution sampling algorithm, stochastic algorithm
- etc...

Mean field limit - McKean-Vlasov SDE

Self-interacting nonlinear diffusion or distribution dependent SDE:

$$dX_t = b_t(X_t, \mu_t)dt + \sigma_t(X_t, \mu_t)dB_t$$

- X_t: state variable with values in $\mathbb{R}^d$
- μ_t: law of X_t
- $(B_t)_{t\geq 0}$: Brownian motion with values in $\mathbb{R}^m$
- coefficients

$$b : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$$
$$\sigma : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{d\times m},$$

where $\mathcal{P}(\mathbb{R}^d)$ is the space of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, endowed with the weak convergence topology.

Known results

- Kac (1956,1958), stochastic toy model
- McKean (1966,1967), non-linear parabolic equations
- Existence and uniqueness of the solutions, see F.Y. Wang (2018-2021), Röckner et al. (2019-2021), Li-Li-Xie (2021), Zhao (2021), Buckdahn-Li-Peng-Rainer (2017),...
- Propagation of chaos: as $N \to \infty$. McKean (1986), Sznitman(1991), Malrieu (2001,2003), Jabin-Wang (2018), Durmus et al. (2020), Liu-Wu-Zhang (2021), Lacker (2021), Delarue-Tse (2021), Guillin et al. (2021), Bao-Huang (2021), Hao-Rockner-Zhang (2022), etc.
- Long time behaviors: as $t \to \infty$. Carrillo-McCann-Villani (2003), Eberle et al. (2016), Luo-Wang (2016), Liu-Wu-Zhang (2021), etc.
- Functional inequalities: Malrieu (2001,2003), Guilin-Liu-Wu-Zhang (2022 AAP), Wang et al. (Harnack inequalities) etc.

Known results

- Large and moderate deviation principles:
  - Empirical measure: $L_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$: Léonard (1987 SPA) , S. Feng (1994), Dupuis et al. (2015$\rightarrow$ 2021 EJP), J. Reygner (2018), Liu-Wu (2020 SPA),
  - Weak interacting diffusions: Dawson-Gärtner (1987), Budhiraja-Dupuis-Fischer (2012), Hoeksema et al. (2020)
  - Freidlin-Wentzell type LDP and MDP for McKean-Vlasov SDEs: Herrmann et al. (2008), Dos Reis et al. (2019), Yuan-suo (2021), etc.
- Central limit theorem Wang-Zhao-Zhu (2021), Yuan-Suo (2021)

Known results - McKean-Vlasov SDEs with jumps

- Liang-Majak-Wang (2021 AIHP) Exponential ergodicity for SDEs and McKeanCVlasov processes with Lévy noise.
- Jourdain-Méléard-Woyczynski (2008 ALEA), Nonlinear SDEs driven by Lévy processes and related PDEs.
- T. Hao and J. Li (2016 NoDEA), Mean-field SDEs with jumps and nonlocal integral-PDEs.
- J. Li (2018SPA), Mean-field forward and backward SDEs with jumps and associated nonlocal quasi-linear integral-PDEs.
- Y. Song (2020 JTP), Gradient estimates and exponential ergodicity for mean-field SDEs with jumps.
- Agarwal-Pagliarani (2021 Stochastics), A Fourier-based Picard-iteration approach for a class of McKean-Vlasov SDEs with Lévy jumps.

Known results - McKean-Vlasov SDEs with jumps

- Andreis-D.Pra-Fischer (2018 SAA), McKean-Vlasov limit for interacting systems with simultaneous jumps.
- Mehri-Scheutzow-Stannat-Zangeneh (2020 AAP), Propagation of chaos for stochastic spatially structured neuronal networks with fully path dependent delays and monotone coefficients driven by jump diffusion noise.
- Erny-Löcherbach-Loukianova (2021 EJP), Conditional propagation of chaos for mean field systems of interacting neurons.
- Erny-Löcherbach-Loukianova (2022+ AAP), Strong error bounds for the convergence to its mean field limit for systems of interacting neurons in a diffusive scaling.

McKean-Vlasov SDEs with jumps

Model:

$$\begin{aligned} \mathrm{d}X_t^\epsilon =& b(t, X_t^\epsilon, \mu_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X_t^\epsilon, \mu_t^\epsilon)\mathrm{d}W(t) \\ &+ \epsilon \int_Z G(t, X_{t-}^\epsilon, \mu_t^\epsilon, z)\widetilde{N}^{\epsilon^{-1}}(\mathrm{d}z, \mathrm{d}t), \ t \in [0, T], \epsilon \in (0, 1], \end{aligned} \tag{1}$$

with initial data $X_0^\epsilon = x$.

- μ_t^ϵ is the law of X_t^ϵ;
- W is a (K-cylindrical) Brownian motion (BM in short);
- N is a PRM on $[0, T] \times Z \times \mathbb{R}_+$ with intensity measure $\text{Leb}_T \otimes \nu \otimes \text{Leb}_\infty$, W and N are mutually independent;
- $N^{\epsilon^{-1}}$ is a Poisson random measure (PRM in short) on $[0, T] \times Z$ with a σ-finite intensity measure $\epsilon^{-1}\mathrm{Leb}_T \otimes \nu$;
- $\widetilde{N}^{\epsilon^{-1}}([0, t] \times B) = N^{\epsilon^{-1}}([0, t] \times B) - \epsilon^{-1} t\nu(B)$, $\forall B \in \mathcal{B}(Z)$ with $\nu(B) < \infty$, is the compensated PRM;
- $N^\varphi((0, t] \times B) = \int_{(0,t] \times B \times \mathbb{R}_+} 1_{[0,\varphi(s,z))}(r)\, N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}r)$.

Asymptotic behaviors under small perturbation

As $\epsilon \to 0$, the solution X^ϵ of (1) will tend to the solution of the following *deterministic* equation:

$$\mathrm{d}X_t^0 = b(t, X_t^0, \mu_t^0)\mathrm{d}t, \quad t \in [0, T], \tag{2}$$

with initial data $X_0^0 = x$.

- $X^0 := \{X_t^0, t \in [0, T]\}$,
- μ_t^0 is the law of X_t^0, i.e. $\mu_t^0 = \delta_{X_t^0}$.

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Let

$$Y_t^\epsilon = \frac{X_t^\epsilon - X_t^0}{\sqrt{\epsilon}\lambda(\epsilon)}, \quad t \in [0, T],$$

- Large deviation principle (LDP): $\lambda(\epsilon) = 1/\sqrt{\epsilon}$;
- Central limit theorem (CLT): $\lambda(\epsilon) = 1$;
- Moderate deviation principle (MDP):

$$\lambda(\epsilon) \to +\infty, \ \ \sqrt{\epsilon}\lambda(\epsilon) \to 0 \text{ as } \epsilon \to 0.$$

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$$\mathrm{d}X_t^0 = b(t, X_t^0, \mu_t^0)\mathrm{d}t, \;\; t \in [0, T], \tag{3}$$

with initial data $X_0^0 = x$.

- $X^0 := \{X_t^0, t \in [0, T]\}$,
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Rate function

Let us recall the definition of a rate function and LDP.

Assume $Y^\epsilon = \{Y^\epsilon(t), t \in [0, T]\} \in D([0, T], H)$ P-a.s.

H: a separable Hilbert space,
$D([0, T], H)$: all càdlàg functions with the Skorokhod topology.

Definition (Rate function)

A function $I : D([0, T], H) \to [0, \infty]$ is called a rate function on $D([0, T], H)$, if for each $M < \infty$, the level set

$$\{x \in D([0, T], H) : I(x) \leq M\}$$

is a *closed* subset of $D([0, T], H)$.
I is said to be good if the level set is *compact*.

Large deviation principle

$$Y^\epsilon(t) = \frac{X^\epsilon(t) - X^0(t)}{\sqrt{\epsilon}\lambda(\epsilon)}, \quad t \in [0, T],$$

Definition (Large deviation principle)

$\{Y^\epsilon\}_{\epsilon>0}$ is said to satisfy a LDP on $D([0, T], H)$ with speed $\frac{1}{\lambda^2(\epsilon)}$ and rate function I if the following two claims hold.

(a) (Upper bound) For each closed subset C of $D([0, T], H)$,

$$\limsup_{\epsilon \to 0} \frac{1}{\lambda^2(\epsilon)} \log P\left(Y^\epsilon \in C\right) \leq - \inf_{x \in C} I(x).$$

(b) (Lower bound) For each open subset O of $D([0, T], H)$

$$\liminf_{\epsilon \to 0} \frac{1}{\lambda^2(\epsilon)} \log P\left(Y^\epsilon \in O\right) \geq - \inf_{x \in O} I(x).$$

- Large deviation principle(LDP): $\lambda(\epsilon) = \frac{1}{\sqrt{\epsilon}}$;
- Moderate deviation principle(MDP):

$$\lambda(\epsilon) \to +\infty, \quad \sqrt{\epsilon}\lambda(\epsilon) \to 0 \text{ as } \epsilon \to 0.$$

Existing results

- Freidlin-Wentzell Large deviation principles for McKean-Vlasov SDEs driven by *Brownian motion*:

  Herrmann, Imkeller, and Peithmann (2008 AOAP), Large deviations and a Kramers' type law for self-stabilizing diffusions.
  Dos Reis, Salkeld, and Tugaut (2019 AOAP), Freidlin-Wentzell LDPs in path space for McKean-Vlasov equations and the functional iterated logarithm law.
  Y. Suo and C. Yuan (2021 Acta. Appl. Math.), Central Limit Theoremand Moderate Deviation Principle for McKean-Vlasov SDEs.
  Adams, Dos Reis, Ravaille, Salkeld and Tugaut (2020 arxiv), Large Deviations and Exit-times for reflected McKean-Vlasov equations with self-stabilizing terms and superlinear drifts.
  W. Liu and L. Wu (2020 SPA), Large deviations for empirical measures of mean-field gibbs measures.
- With jumps: Y. Cai, J. Huang and V. Maroulas (2015 Statist. Probab. Lett.), Large deviations of mean-field stochastic differential equations with jumps.

Existing results - continued

Consider

$$\mathrm{d}X^\epsilon(t) = b(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}W(t), \ \ t \in [0, T], \epsilon \in (0, 1], \tag{4}$$

with initial data $X^\epsilon(0) = x$.

- μ_t^ϵ is the law of $X^\epsilon(t)$,
- W is a Brownian motion (BM in short).

As $\epsilon \to 0$,

$$\mathrm{d}X^0(t) = b(t, X^0(t), \mu_t^0)\mathrm{d}t, \ \ t \in [0, T], \tag{5}$$

with initial data $X^0(0) = x$.

- μ_t^0 is the law of $X^0(t)$.

Set

$$\mathrm{d}Y^\epsilon(t) = b(t, Y^\epsilon(t), \mu_t^0)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, Y^\epsilon(t), \mu_t^0)\mathrm{d}W(t), \ \ t \in [0, T], \epsilon \in (0, 1], \tag{6}$$

with initial data $Y^\epsilon(0) = x$.

Existing results - continued

$$\mathrm{d}X^\epsilon(t) = b(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}W(t)$$

$$\mathrm{d}Y^\epsilon(t) = b(t, Y^\epsilon(t), \mu_t^0)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, Y^\epsilon(t), \mu_t^0)\mathrm{d}W(t)$$

Strategies:

- Step 1: LDP for Y^ϵ as $\epsilon \to 0$
  - Discretization, approximation and exponential equivalence arguments (requiring strong conditions): *Herrmann-Imkeller-Peithmann (2008), Reis-Salkeld-Tugaut (2019), Adams-Reis-Ravaille-Salkeld-Tugaut (2020)*
  - Weak convergence method: Y. Suo and C. Yuan (2021, with Lion's derivatives)
- Step 2: X^ϵ and Y^ϵ are exponentially equivalent as $\epsilon \to 0$.

Existing results - continued

$$\mathrm{d}X^\epsilon(t) = b(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}W(t)$$

Stronger conditions on the coefficients are required to use exponential approximation arguments.

- Herrmann-Imkeller-Peithmann (2008): Additive noise
- Reis-Salkeld-Tugaut (2019): Multiplicative noise
  - b: monotone growth condition, locally Lipschitz with polynomial growth in x, Lipschitz in μ; σ: bounded Lipschitz
  - there exists $\beta \in (0, 1]$ such that

    $$|\sigma(s, y, \mu) - \sigma(t, y, \mu)| \leq L|s - t|^\beta, \quad |b(s, y, \mu) - b(t, y, \mu)| \leq L|s - t|^\beta.$$
- Adams-Reis-Ravaille-Salkeld-Tugaut (2020)
  - reflected McKean-Vlasov SDEs, multiplicative noise
  - there exists $\beta \in (0, 1]$ such that

    $$|\sigma(s, y, \mu) - \sigma(t, y, \mu)| \leq L|s - t|^\beta.$$
- Y. Suo and C. Yuan (2021)

Weak convergence method

- **The exponential approximation arguments** are not suitable to deal with *SDE with jumps* and *SPDEs*.
- **The weak convergence method** is proved to be a powerful tool to establish large and moderate deviation principles for various dynamical systems driven by Gaussian noise and/or PRM.
- **The aim**: Fully apply the weak convergence method to establish large and moderate deviation principles for *McKean-Vlasov SDEs/SPDEs with jumps/delay/memory*...

✓ A. Budhiraja, P. Dupuis, A. Ganguly (2016), Moderate deviation principles for stochastic differential equations with jumps. *Ann. Probab.* **44**, 1723-1775.

✓ A. Budhiraja, P. Dupuis, and V. Maroulas, (2011) Variational representations for continuous time processes. *Ann. Inst. Henri Poincaré, Probab. Stat.*, **47**, 725-747.

✓ A. Budhiraja, P. Dupuis, (2000) A variational representation for positive functionals of an infinite dimensional Brownian motion. *Probab. Math. Stat.* **20**, 39-61.

✓ A. Budhiraja, P. Dupuis, V. Maroulas,(2008) Large deviations for infinite dimensional stochastic dynamical systems continuous time processes. *Ann. Probab.* **36** 1390-1420.

A. Budhiraja, J. Chen, and P. Dupuis, (2013) Large deviations for stochastic partial differential equations driven by a Poisson random measure, *Stoch. Proc. Appl.*, **123**, 523-560.

Z. Dong, J. Xiong, J. Zhai, T. Zhang, (2017) A moderate deviation principle for 2-D stochastic Navier-Stokes equations driven by multiplicative Lévy noises. *J. Funct. Anal.*, **272**, 227-254.

Z. Dong, J. Wu, R. Zhang and T. Zhang, (2020) Large deviation principles for first-order scalar conservation laws with stochastic forcing. *Ann. Appl. Probab.*, **30**, no. 1, 324-367.

Weak convergence method

Consider large deviation principles for X^ϵ as the parameter ϵ tends to 0,

$$\mathrm{d}X^\epsilon(t) = b(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}W(t), \ \ t \in [0, T], \epsilon \in (0, 1],$$

with initial data $X^\epsilon(0) = x$.

- μ_t^ϵ is the law of $X^\epsilon(t)$,
- W is a Brownian motion (BM in short).

The basic step is to find the mapping Γ^ε such that

$$X^\epsilon = \Gamma^\varepsilon(W(\cdot)),$$

Weak convergence method

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with initial data $X^\epsilon(0) = x$.

- μ_t^ϵ is the law of $X^\epsilon(t)$,
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The basic step is to find the mapping Γ^ε such that

$$X^\epsilon = \Gamma^\varepsilon(W(\cdot)),$$

and then to identify the correct equation satisfied by

$$X^{\epsilon, h^\varepsilon} := \Gamma^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot \dot{h}^\varepsilon(s)ds\right).$$

Weak convergence method

Set

$$\mathrm{d}X^0(t) = b(t, X^0(t), \mu_t^0)\mathrm{d}t, \quad t \in [0, T], \tag{7}$$

with initial data $X^0(0) = x$.

- $X^0 := \{X^0(s), s \in [0, T]\}$,
- μ_t^0 is the law of $X^0(t)$.

$X^{0,h} := \Gamma^0 \left(\int_0^{\cdot} \dot{h}(s) ds \right)$ is the solution to the following controlled ODE:

$$\mathrm{d}X^{0,h}(t) = b(t, X^{0,h}(t), \mu_t^0)\mathrm{d}t + \sigma(t, X^{0,h}(t), \mu_t^0)\dot{h}(t)\mathrm{d}t, \quad t \in [0, T], \tag{8}$$

where μ_t^0 is the distribution of $X^0(t)$.

Weak convergence method

For each $f \in L^2([0,T],K)$, we introduce the quantity

$$Q_1(f) := \frac{1}{2}\int_0^T \|f(s)\|_K^2 \,\mathrm{d}s,$$

and for each $m > 0$, denote

$$S_1^m := \Big\{ f \in L^2([0,T],K) : \; Q_1(f) \le m \Big\}.$$

Equipped with the weak topology, S_1^m is a compact subset of $L^2([0,T],K)$.

- $\{\dot{h}_n, n \ge 1\} \in S_1^m$ and $\dot{h} \in S_1^m$:

$$\dot{h}_n \to \dot{h} \quad \text{in } S_1^m.$$

- $\Gamma^0\left(\int_0^\cdot \dot{h}_n(s)ds\right) \to \Gamma^0\left(\int_0^\cdot \dot{h}(s)ds\right)$ in $D([0,T],H)$.

Weak convergence method

For any $m \in (0, \infty)$, let $\mathcal{S}_1^m$ be a space of stochastic processes on Ω defined by

$$\mathcal{S}_1^m := \{\varphi\colon [0,T] \times \Omega \to K : \mathbb{F}\text{-predictable and } \varphi(\cdot,\omega) \in S_1^m \text{ for } P\text{-a.e. } \omega \in \Omega\}.$$

- $\{\dot{h}^\varepsilon, \varepsilon > 0\} \subset \mathcal{S}_1^m$ and $\dot{h} \in \mathcal{S}_1^m$:

$$\lim_{\varepsilon \to 0} \dot{h}^\varepsilon = \dot{h} \text{ in law as } S_1^m\text{-valued random elements}$$

- $\Gamma^\epsilon(W + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \dot{h}^\varepsilon(s)ds) \to \Gamma^0 \left(\int_0^\cdot \dot{h}(s)ds\right)$ in law as $D([0,T], H)$-valued random elements

Weak convergence method

We only need to check

- $\Gamma^0\left(\int_0^\cdot \dot{h}_n(s)ds\right) \to \Gamma^0\left(\int_0^\cdot \dot{h}(s)ds\right)$ in $D([0,T],H)$.
- $\Gamma^\epsilon(W + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot \dot{h}^\varepsilon(s)ds) - \Gamma^0(\int_0^\cdot \dot{h}^\varepsilon(s)ds) \to 0$ in probability as $D([0,T],H)$-valued random elements

- A. Matoussi, W. Sabbagh, and T. Zhang, Large deviation principle of obstacle problems for Quasilinear Stochastic PDEs. *Appl. Math. Optim.*, https://doi.org/10.1007/s00245-019-09570-5.
- W. Hong, S. Li, W. Liu, Large deviation principle for McKean-Vlasov Quasilinear stochastic stochastic evolution equations, arXiv:2103.11398

Controlled McKean-Vlasov SDE

Assume that there is a unique strong solution X^ϵ.

$$\mathrm{d}X^\epsilon(t) = b(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^\epsilon(t), \mu_t^\epsilon)\mathrm{d}W(t), \ \ t \in [0, T], \epsilon \in (0, 1],$$

with initial data $X^\epsilon(0) = x$. μ_t^ϵ is the law of $X^\epsilon(t)$.
Then, there exists a measurable map Γ^ϵ such that the solution X^ϵ can be represented as

$$X^\epsilon = \Gamma^\epsilon(W(\cdot)).$$

$X^{\epsilon,h^\varepsilon} := \Gamma^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot \dot{h}^\varepsilon(s)ds\right)$??? is the solution to the following controlled SDE:

$$\begin{aligned} \mathrm{d}X^{\epsilon,h^\varepsilon}(t) &= b(t, X^{\epsilon,h^\varepsilon}(t), \mu_t^{\epsilon,h^\varepsilon})\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^{\epsilon,h^\varepsilon}(t), \mu_t^{\epsilon,h^\varepsilon})\mathrm{d}W(t) \\ &\quad + \sigma(t, X^{\epsilon,h^\varepsilon}(t), \mu_t^{\epsilon,h^\varepsilon})\dot{h}^\varepsilon(t)\mathrm{d}t, \quad t \in [0, T], \end{aligned} \tag{9}$$

where $\mu_t^{\epsilon,h_t^\varepsilon}$ is the distribution of $X^{\epsilon,h^\varepsilon}(t)$.

Y. Cai, J. Huang and V. Maroulas, Large deviations of mean-field stochastic differential equations with jumps. *Statist. Probab. Lett.*, **96** (2015), 1-9.

Controlled McKean-Vlasov SDE

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with initial data $X^\epsilon(0) = x$. μ_t^ϵ is the law of $X^\epsilon(t)$.
Then, there exists a measurable map Γ^ϵ such that the solution X^ϵ can be represented as

$$X^\epsilon = \Gamma^\epsilon(W(\cdot)).$$

In fact, $X^{\epsilon, h^\varepsilon} := \Gamma^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \dot{h}^\varepsilon(s)ds\right)$ is the solution to the following controlled SDE:

$$\begin{aligned}\mathrm{d}X^{\epsilon, h^\varepsilon}(t) &= b(t, X^{\epsilon, h^\varepsilon}(t), \mu_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^{\epsilon, h^\varepsilon}(t), \mu_t^\epsilon)\mathrm{d}W(t) \\ &\quad + \sigma(t, X^{\epsilon, h^\varepsilon}(t), \mu_t^\epsilon)\dot{h}^\varepsilon(t)\mathrm{d}t, \quad t \in [0, T], \end{aligned} \tag{10}$$

where μ_t^ϵ is the distribution of $X^\epsilon(t)$.

Controlled McKean-Vlasov SDE: example

Consider the following simple one dimensional MVSDE:

$$X^\epsilon(t) = x_0 + \int_0^t \mathbb{E}(X^\epsilon(s))\mathrm{d}s + \sqrt{\epsilon}W(t). \tag{11}$$

There exists a map Γ^ϵ such that $X^\epsilon = \Gamma^\epsilon(W)$.

$Y^\epsilon := \Gamma^\epsilon(W + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \dot{h}^\varepsilon(s)ds)$?:

- $\mathbb{E}(X^\epsilon(t)) = x_0 + \int_0^t \mathbb{E}(X^\epsilon(s))\mathrm{d}s$. Hence $\mathbb{E}(X^\epsilon(t)) = x_0 e^t$.
- Thus

$$X^\epsilon(t) = x_0 + \int_0^t x_0 e^s \mathrm{d}s + \sqrt{\epsilon}W(t) = \Gamma^\epsilon(W)(t). \tag{12}$$

Therefore, Y^ϵ is the solution of the equation:

$$\begin{aligned} Y^\epsilon(t) =& x_0 + \int_0^t x_0 e^s \mathrm{d}s + \sqrt{\epsilon}W(t) + \int_0^t \dot{h}^\varepsilon(s)\mathrm{d}s \\ =& x_0 + \int_0^t \mathbb{E}(X^\epsilon(s))\mathrm{d}s + \sqrt{\epsilon}W(t) + \int_0^t \dot{h}^\varepsilon(s)\mathrm{d}s. \end{aligned} \tag{13}$$

Y^ϵ **does NOT satisfy the following controlled SDE**:

$$Y^\epsilon(t) = x_0 + \int_0^t \mathbb{E}(Y^\epsilon(s))\mathrm{d}s + \sqrt{\epsilon}W(t) + \int_0^t \dot{h}^\varepsilon(s)\mathrm{d}s. \tag{14}$$

General framework

We consider the following general distribution-dependent SDEs with jumps

$$\mathrm{d}Y(t) = b(t, Y, J)\mathrm{d}t + \sigma(t, Y, J)\mathrm{d}W(t) + \int_Z G(t, Y, J, z)\widetilde{N}^1(\mathrm{d}z, \mathrm{d}t) \quad (15)$$

with initial value $Y(0) = h \in H$.

Remark: This abstract formulation of SDEs (15) is general enough to cover many types of SPDEs, such as SPDEs with delay, DDSPDEs, MKSDEs, etc.

We now introduce some definitions related to the solutions of (15).

Definition

For a fixed $J \in Pr(D([0, T], H))$, Y is called a solution of (15) if

(a) $Y = \{Y(t), t \in [0, T]\}$ is an adapted process,

(b) $\int_0^T \|b(t, Y, J)\|_E \mathrm{d}t + \int_0^T \|\sigma(t, Y, J)\|^2_{\mathcal{L}_2} \mathrm{d}t + \int_0^T \int_Z \|G(t, Y, J, z)\|^2_H \nu(\mathrm{d}z)\mathrm{d}t < \infty$, P-a.s.,

(c)

$$\begin{aligned} Y(t) =& h + \int_0^t b(s, Y, J)\mathrm{d}s + \int_0^t \sigma(s, Y, J)\mathrm{d}W(s) \\ &+ \int_0^t \int_Z G(s, Y, J, z)\widetilde{N}^1(\mathrm{d}z, \mathrm{d}s), \ t \in [0, T], \ P\text{-a.s.} \end{aligned}$$

General framework - continued

Definition (Pathwise uniqueness)

The pathwise uniqueness is said to hold for (15) with the fixed $J \in Pr(D([0, T], H))$, if for any two solutions Y_1 and Y_2 of (15),

$$Y_1(t) = Y_2(t), \;\; t \in [0, T], \;\; P\text{-a.s.}$$

- Now consider the McKean-Vlasov equation:

$$\begin{aligned} \mathrm{d}X(t) =& b(t, X, Law(X))\mathrm{d}t + \sigma(t, X, Law(X))\mathrm{d}W(t) \\ &+ \int_Z G(t, X, Law(X), z)\widetilde{N}^1(\mathrm{d}z, \mathrm{d}t) \end{aligned} \tag{16}$$

- $X = \{X(t)\}_{0\le t\le T}$ is a solution to equation (16) if X is a solution of (15) with $J = Law(X)$.
- The pathwise uniqueness is said to hold if for any two solutions X_1 and X_2 of (16),

$$X_1(t) = X_2(t), \;\; t \in [0, T] \; P\text{-a.s.},$$

and hence $Law(X_1) = Law(X_2)$.

General framework - continued

Theorem

Fix $J \in Pr(D([0,T],H))$. Suppose that Y is a solution of (15), and pathwise uniqueness holds with the fixed J. Then there exists a unique map Γ_J such that

$$Y = \Gamma_J(W, N^1).$$

Moreover for any $m \in (0,\infty)$ and $u = (\phi, \psi) \in \mathcal{S}_1^m \times \mathcal{S}_2^m$, let

$$Y^u := \Gamma_J(W + \int_0^{\cdot} \phi(s)\mathrm{d}s, N^{\psi}), \tag{17}$$

then we have

(a) $Y^u = \{Y^u(t), t \in [0,T]\}$ *is an adapted process,*

(b)

$$\int_0^T \|b(t,Y^u,J)\|_E \mathrm{d}t + \int_0^T \|\sigma(t,Y^u,J)\|_{\mathcal{L}_2}^2 \mathrm{d}t + \int_0^T \|\sigma(t,Y^u,J)\phi(t)\|_H \mathrm{d}t$$
$$+ \int_0^T \int_Z \|G(t,Y^u,J,z)\|_H^2 \psi(t,z)\nu(\mathrm{d}z)\mathrm{d}t + \int_0^T \int_Z \|G(t,Y^u,J,z)(\psi(t,z) - 1)\|_H$$
$$< \infty, \quad P\text{-a.s.}$$

General framework - continued

(c) as a stochastic equation on E, Y^u satisfies

$$\begin{aligned} Y^u(t) =& h + \int_0^t b(s, Y^u, J)\mathrm{d}s + \int_0^t \sigma(s, Y^u, J)\mathrm{d}W(s) + \int_0^t \sigma(s, Y^u, J)\phi(s)\mathrm{d}s \\ &+ \int_0^t \int_Z G(s, Y^u, J, z)\Big(N^\psi(\mathrm{d}z, \mathrm{d}s) - \nu(\mathrm{d}z)\mathrm{d}s\Big), \ t \in [0, T], \ P\text{-a.s.} \end{aligned} \tag{18}$$

Moreover, Y^u is the unique stochastic process satisfying (a)-(c).

Remark: Although the claim in previous Theorem is not surprising, its rigorous proof requires the careful use of the Girsanov Theorem for the mixture of Brownian motion and Poisson random measures.

General framework - continued

Theorem

Assume that X is a solution of (16) with initial value $X(0) = h \in H$, and that the pathwise uniqueness holds for (15) with $J = Law(X)$.
Then

$$X = \Gamma_{Law(X)}(W, N^1)$$

where $\Gamma_{Law(X)}$ is the map Γ_J with $J = Law(X)$.
Moreover for any $m \in (0, \infty)$ and $u = (\phi, \psi) \in \mathcal{S}_1^m \times \mathcal{S}_2^m$, let $X^u := \Gamma_{Law(X)}(W + \int_0^{\cdot} \phi(s)\mathrm{d}s, N^{\psi})$, then we have

(a) $X^u = \{X^u(t), t \in [0, T]\}$ *is an adapted process,*

(b)

$$\begin{aligned} X^u(t) =& h + \int_0^t b(s, X^u, Law(X))\mathrm{d}s + \int_0^t \sigma(s, X^u, Law(X))\mathrm{d}W(s) \\ &+ \int_0^t \sigma(s, X^u, Law(X))\phi(s)\mathrm{d}s + \int_0^t \int_Z G(s, X^u, Law(X), z)\widetilde{N}^{\psi}(\mathrm{d}z, \mathrm{d}s) \\ &+ \int_0^t \int_Z G(s, X^u, Law(X), z)\Big(\psi(s, z) - 1\Big)\nu(\mathrm{d}z)\mathrm{d}s, \ \ t \in [0, T] \ P\text{-a.s.} \end{aligned} \quad (19)$$

Applications: conditions for LDP

$$
\begin{aligned}
X^\epsilon(t) =& h + \int_0^t b_\epsilon(s, X^\epsilon(s), Law(X^\epsilon(s)))\mathrm{d}s + \sqrt{\epsilon}\int_0^t \sigma_\epsilon(s, X^\epsilon(s), Law(X^\epsilon(s)))\mathrm{d}W(s) \\
&+ \epsilon \int_0^t \int_Z G_\epsilon(s, X^\epsilon(s-), Law(X^\epsilon(s)), z)\widetilde{N}^{\epsilon^{-1}}(\mathrm{d}z, \mathrm{d}s), \ \ t \in [0, T],
\end{aligned}
$$

We assume that (A0) For any $\epsilon > 0$, there exists a unique solution X^ϵ.
There are $L > 0$ and $q \geq 1$ such that for each $t \in [0, T]$,

(A1)

$$
\begin{aligned}
&\langle x - x', b(t, x, \mu) - b(t, x', \mu)\rangle \leq L|x - x'|^2, \\
&|b(t, x, \mu) - b(t, x, \mu')| \leq LW_2(\mu, \mu'), \\
&|b(t, x, \mu) - b(t, x', \mu)| \leq L(1 + |x|^{q-1} + |x'|^{q-1})|x - x'|, \\
&\|\sigma(t, x, \mu) - \sigma(t, x', \mu')\|_{\mathcal{L}_2} \leq L(|x - x'| + W_2(\mu, \mu')), \\
&\int_0^T \left(|b(t, 0, \delta_0)| + \|\sigma(t, 0, \delta_0)\|^2_{\mathcal{L}_2}\right) \mathrm{d}t < \infty.
\end{aligned}
$$

Applications: conditions for LDP

(A2) There exist $L_1, L_2, L_3 \in \mathcal{H} \cap L^2(\nu)$ such that for all $t \in [0, T]$, $x,\ x' \in R^d$, $\mu,\ \mu' \in \mathcal{P}_2$ and $z \in Z$,

$$|G(t,x,\mu,z) - G(t,x',\mu',z)| \leq L_1(z)\left(|x-x'| + W_2(\mu,\mu')\right),$$

$$|G(t,0,\delta_0,z)| \leq L_2(z).$$

and there exists nonnegative constant $\varrho_{G,\epsilon}$ converging to 0 such that

$$\sup_{(t,x,\mu)\in[0,T]\times R^d\times\mathcal{P}_2} |G_\epsilon(t,x,\mu,z) - G(t,x,\mu,z)| \leq \varrho_{G,\epsilon} L_3(z).$$

(A3) As $\epsilon \downarrow 0$, the maps b_ϵ and σ_ϵ converge uniformly to b and σ respectively, that is, there exist nonnegative constants $\varrho_{b,\epsilon}$ and $\varrho_{\sigma,\epsilon}$ converging to 0 as $\epsilon \downarrow 0$ such that

$$\sup_{(t,x,\mu)\in[0,T]\times R^d\times\mathcal{P}_2} \left(|b_\epsilon(t,x,\mu) - b(t,x,\mu)|\right) \leq \varrho_{b,\epsilon}, \qquad (20)$$

$$\sup_{(t,x,\mu)\in[0,T]\times R^d\times\mathcal{P}_2} \left(\|\sigma_\epsilon(t,x,\mu) - \sigma(t,x,\mu)\|_{\mathcal{L}_2}\right) \leq \varrho_{\sigma,\epsilon}. \qquad (21)$$

(A4) Pathwise uniqueness holds with $J = Law(X^\epsilon)$.

Applications: LDP

For each measurable function $g\colon [0,T]\times Z \to [0,\infty)$, define

$$Q_2(g) := \int_{[0,T]\times Z} \ell(g(s,z))\,\nu(\mathrm{d}z)\mathrm{d}s,$$

where $\ell(x) = x\log x - x + 1$, $\ell(0) := 1$. For each $m > 0$, denote

$$S_2^m := \Big\{ g : [0,T]\times Z \to [0,\infty) : \; Q_2(g) \le m \Big\}.$$

Any measurable function $g \in S_2^m$ can be identified with a measure $\hat{g} \in M_{FC}([0,T]\times Z)$, defined by

$$\hat{g}(A) = \int_A g(s,z)\,\nu(\mathrm{d}z)\mathrm{d}s, \;\; \forall A \in \mathcal{B}([0,T]\times Z). \tag{22}$$

This identification induces a topology on S_2^m under which S_2^m is a compact space (see the work of Budhiraja-Chen-Dupuis).

Applications: LDP main result

Theorem

The solutions $\{X^\epsilon\}$ *satisfy the LDP on* $D([0,T],\mathbb{R}^d)$ *with speed* ϵ *and the rate function* I *given by*

$$I(g) := \inf\{Q_1(\phi) + Q_2(\psi) : u = (\phi,\psi) \in S,\ \ Y^u = g\},\ \ g \in D([0,T],\mathbb{R}^d), \tag{23}$$

where for $u = (\phi,\psi) \in S$, Y^u *is the unique solution to the following equation:*

$$Y^u(t) = h + \int_0^t b(s, Y^u(s), Law(X^0(s)))\mathrm{d}s + \int_0^t \sigma(s, Y^u(s), Law(X^0(s)))\phi(s)\mathrm{d}s \tag{24}$$

$$+ \int_0^t \int_Z G(s, Y^u(s), Law(X^0(s)), z)(\psi(s,z) - 1)\nu(\mathrm{d}z)\mathrm{d}s, \quad t \in [0,T]. \tag{25}$$

Here we use the convention that the infimum of an empty set is ∞.

Applications: conditions for MDP

For any $t \in [0, T]$ and $\mu \in \mathcal{P}_2$, let $b_2'(t, x, \mu)$ denote the derivative of $b(t, x, \mu)$ with respect to the variable x.
In order to obtain the MDP, we give the following additional assumptions.

(B1) There are $L', q' \geq 0$ such that for each $x, x' \in R^d$,

$$|b_2'(s, x, Law(X^0(s))) - b_2'(s, x', Law(X^0(s)))| \tag{26}$$

$$\leq L'(1 + |x|^{q'} + |x'|^{q'})|x - x'|, \tag{27}$$

and

$$\int_0^T |b_2'(t, X^0(t), Law(X^0(t)))| \mathrm{d}t < \infty.$$

(B2)

$$\lim_{\epsilon \to 0} \frac{\varrho_{b,\epsilon}}{\lambda(\epsilon)} = 0$$

where $\varrho_{b,\epsilon}$ is given in (A3).

Applications: MDP main result

Theorem

Then $\{M^\epsilon := \frac{1}{\lambda(\epsilon)}(X^\epsilon(t) - X^0(t))\}$ *satisfies a LDP on* $D([0,T],\mathbb{R}^d)$ *with speed* $\epsilon/\lambda^2(\epsilon)$ *and the rate function* I *given by for any* $g \in D([0,T],\mathbb{R}^d)$

$$I(g) := \inf_{\{u=(\phi,\varphi)\in L^2([0,T],\mathbb{R}^d)\times L_2(\nu_T), K^u = g\}} \left\{ \frac{1}{2}\int_0^T |\phi(s)|^2 \mathrm{d}s + \frac{1}{2}\int_0^T\int_Z |\varphi(s,z)|^2 \nu(\mathrm{d}z)\mathrm{d}s \right\}$$

where for $u = (\phi,\varphi) \in L^2([0,T],\mathbb{R}^d) \times L_2(\nu_T)$*,* K^u *is the unique solution of the following equation*

$$\begin{cases} \mathrm{d}K^u(t) = b_2'(t, X^0(t), Law(X^0(t)))K^u(t)\mathrm{d}t + \sigma(t, X^0(t), Law(X^0(t)))\phi(t)\mathrm{d}t \\ \qquad\qquad + \int_Z G(t, X^0(t), Law(X^0(t)), z)\varphi(t,z)\nu(\mathrm{d}z)\mathrm{d}t \\ K^u(0) = 0. \end{cases} \tag{28}$$

Thanks for your kind attention!